

# Center-focus determination and limit cycles bifurcation for $p : q$ homogeneous weight singular point \*

Tao Liu

*School of Mathematics, Central South University, Changsha, Hunan, 410083, P.R. China*

Feng Li

*School of Science, Linyi University, Linyi, Shandong, 276005, P.R. China*

Yirong Liu

*School of Mathematics, Central South University, Changsha, Hunan, 410083, P.R. China*

Shimin Li

*School of Mathematics and Statistics, Guangdong University of Finance and Economics,  
Guangzhou, Guangdong, 510320, P.R. China*

## Abstract

The quasi-homogeneous (and in general non-homogeneous) polynomial differential systems have been studied from many different points of view. In this paper, Center-focus determination and limit cycles bifurcation for  $p : q$  homogeneous weight singular point are investigated. Some prosperities of Successive function and focus values are discussed, furthermore, the method of computing focal values is given. As an example, Center-focus determination and limit cycle bifurcation for  $2 : 3$  homogeneous weight singular point are studied, three or five limit cycles in the neighborhood of origin can be obtained by different perturbations.

*MSC:* 34C05; 34C07

*Keywords:* Degenerate critical point, limit cycle, center, homogeneous weight singular point.

---

\*This research was supported by the National Natural Science Foundation of China (11371373)

# 1 Introduction

It is well known that center-focus determination is difficult and important in qualitative theory of planar system. It is far from being solved although there were many results for elementary singular point and nilpotent singular point [1]. Hopf bifurcation has also been investigated intensively because it is closely related to center-focus problem and the 16th problem of Hilbert. For planar ordinary differential equations, there were many good results for planar systems. For example, one of the best-known results was  $M(2) = 3$  [2] for a planar system with an elementary critical point. Here,  $M(n)$  denotes the maximal number of small-amplitude limit cycles around a singular point with  $n$  being the degree of polynomials in the vector field. When  $n = 3$ , the authors constructed two different cubic systems to show there exist 9 limit cycles for cubic systems in [3] and [4]. Recently, Yu and Tian showed that there could be twelve limit cycles around a singular point in a planar cubic-degree polynomial system[5]. But for a system with a degenerate critical points it is still a hard work to solve its center problem and to determine the number of limit cycles. When a critical point is degenerate, its center problem has also been investigated by many authors, see [6, 7, 8, 9, 10, 11]. There were also many results about the bifurcation of limit cycles [12, 13, 14, 15], for more detail, see [16, 17]. A special system with total degenerate critical point was investigated by Liu etc. in [18].

Some special systems have also been investigated. The homogeneous polynomial differential systems have been studied by several authors. Thus, the quadratic homogeneous ones by [29, 34, 40, 41, 42, 43, 45]; the cubic homogeneous ones by [27]; the homogeneous systems of arbitrary degree by [25, 27, 28, 35], and others. In these previous papers is described an algorithm for studying the phase portraits of homogeneous polynomial vector fields for all degree, the classification of all phase portraits of homogeneous polynomial vector fields of degree 2 and 3, the algebraic classifications of homogeneous polynomial vector fields and the characterization of structurally stable homogeneous polynomial vector fields.

The quasi-homogeneous (and in general non-homogeneous) polynomial differential systems have been studied from many different points of view, mainly for their integrability [21, 22, 30, 31, 32, 33, 38], for their rational integrability [23, 46, 47, 48], for their polynomial integrability [26, 39, 44], for their centers [19, 20, 36], for their normal forms [24], for their limit cycles [37], ... . But up to now there was not an algorithm for constructing all the quasi-homogeneous polynomial differential systems of a given degree. Han and xiong classified all centers of a class of quasi-homogeneous polynomial differential systems of degree 5 in [49]. The same authors investigated a class of quasi-homogeneous polynomial systems with a given weight degree in [50].

The cyclicity and center problems are studied for some subfamilies of semi-quasihomogeneous polynomial systems by Zhao in [51].

In this paper, center-focus determination and limit cycle bifurcation for  $p : q$  homogeneous weight singular point will be investigated. In section two, homogeneous weight system and generalized polar coordinate are given; Some prosperities of successive function and focus values are discussed in section three; As application, center-focus determination and limit cycle bifurcation for  $2 : 3$  homogeneous weight singular point are investigated in section four.

## 2 Homogeneous weight system and generalized Polar Coordinate

In this section, some necessary definitions are given.

**Definition 2.1.** *If there exist positive integer  $p, q, m$ , which satisfy*

$$F(\lambda^p x, \lambda^q y) \equiv \lambda^m F(x, y), \quad (2.1)$$

*then  $F(x, y)$  is called to be  $m$ -order homogeneous weight function of  $x, y$  with weight  $p, q$ .*

It is easy to testify that if  $p, q, m$  are positive integers and  $F(x, y)$  is a  $m$ -order homogeneous weight polynomial function of  $x, y$  with weight  $p, q$ , then  $F(x, y)$  can be written as

$$F(x, y) = \sum_{kp+jq=m} C_{kj} x^k y^j. \quad (2.2)$$

Considering the following system

$$\frac{dx}{dt} = -\lambda_1 y^{2p-1} + \sum_{k+j=2}^{\infty} a_{kj} x^k y^j, \quad \frac{dy}{dt} = \lambda_2 x^{2q-1} + \sum_{k+j=2}^{\infty} b_{kj} x^k y^j, \quad (2.3)$$

where  $p, q$  are positive integer,  $a_{0,2p-1} = b_{2q-1,0} = 0$ ,  $\lambda_1 > 0, \lambda_2 > 0$ . Suppose functions of right hand of system (2.3) are power series of  $x, y$  with non-zero radius convergence.

Without loss of generality, let

$$\lambda_1 = p, \quad \lambda_2 = q, \quad (2.4)$$

otherwise, let

$$x = \sqrt[q]{\frac{q}{\lambda_2}} u, \quad y = \sqrt[p]{\frac{p}{\lambda_1}} v, \quad \frac{dt}{d\tau} = \sqrt[p]{\frac{p}{\lambda_1}} \sqrt[q]{\frac{q}{\lambda_2}}. \quad (2.5)$$

When (2.4) holds, system (2.3) could be rewritten as

$$\frac{dx}{dt} = -py^{2p-1} + \sum_{k+j=2}^{\infty} a_{kj} x^k y^j, \quad \frac{dy}{dt} = qx^{2q-1} + \sum_{k+j=2}^{\infty} b_{kj} x^k y^j, \quad (2.6)$$

The origin of system (2.6) is an elementary singular point when  $p = q = 1$  and a nilpotent singular points when  $p = 1, q > 1$  (or  $p > 1, q = 1$ ), there are many differential topological constructions of phase curves in the neighborhood of origin of system (2.6), see [?]. When  $p > 1, q > 1$ , topological constructions of phase curves in the neighborhood of origin of system (2.6) has not been investigated completely.

In this paper, we do not consider the topological constructions of phase curves in the neighborhood of origin of system (2.6). For sufficiently small  $h > 0$ , the solution of system (2.6) which satisfy initial condition  $x|_{t=0} = h^p, y|_{t=0} = 0$  goes around the origin at the neighborhood of  $x^{2q} + y^{2p} = h^{2pq}$ , then phase curves in the neighborhood of origin of system (2.6) can be studied by transformations

$$x = r^p \cos \theta, \quad y = r^q \sin \theta \quad (2.7)$$

So the system could be written as

$$\begin{aligned} \frac{dx}{dt} &= -py^{2p-1} + \sum_{\substack{kp+jq> \\ (2p-1)q}}^{\infty} a_{kj}x^k y^j = \mathcal{X}(x, y), \\ \frac{dy}{dt} &= qx^{2q-1} + \sum_{\substack{kp+jq> \\ (2q-1)p}}^{\infty} b_{kj}x^k y^j = \mathcal{Y}(x, y). \end{aligned} \quad (2.8)$$

**Definition 2.2.** system (2.8) is called to be  $p : q$  homogeneous weight system, and the origin of system (2.8) is called to be homogeneous weight focus (or center) with weight  $p : q$ .

**Example 2.1.** Homogeneous weight system with weight  $2 : 3$  can be written as

$$\begin{aligned} \frac{dx}{dt} &= -2y^3 + \sum_{2k+3j=10}^{\infty} a_{kj}x^k y^j \\ &= -2y^3 + (a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4) + \sum_{k+j=5}^{\infty} a_{kj}x^k y^j, \\ \frac{dy}{dt} &= 3x^5 + \sum_{2k+3j=11}^{\infty} b_{kj}x^k y^j = 3x^5 + (b_{13}xy^3 + b_{04}y^4) \\ &\quad + (b_{41}x^4y + b_{32}x^3y^2 + b_{23}x^2y^3 + b_{14}xy^4 + b_{05}y^5) + \sum_{k+j=6}^{\infty} b_{kj}x^k y^j. \end{aligned} \quad (2.9)$$

Now, the functions of right hand of system (2.8) can be written as a homogeneous weight polynomial power series

$$\begin{aligned} \frac{dx}{dt} &= -py^{2p-1} + \sum_{m=2pq-q+1}^{\infty} \mathcal{X}_m(x, y) = \mathcal{X}(x, y), \\ \frac{dy}{dt} &= qx^{2q-1} + \sum_{m=2pq-p+1}^{\infty} \mathcal{Y}_m(x, y) = \mathcal{Y}(x, y), \end{aligned} \quad (2.10)$$

where

$$\mathcal{X}_m(x, y) = \sum_{kp+jq=m} a_{kj} x^k y^j, \quad \mathcal{Y}_m(x, y) = \sum_{kp+jq=m} b_{kj} x^k y^j \quad (2.11)$$

are  $m$ -order homogeneous weight polynomial of  $x, y$  with weight  $p, q$  which satisfy

$$\begin{aligned} \mathcal{X}_m(r^p \cos \theta, r^q \sin \theta) &= r^m \mathcal{X}_m(\cos \theta, \sin \theta), \\ \mathcal{Y}_m(r^p \cos \theta, r^q \sin \theta) &= r^m \mathcal{Y}_m(\cos \theta, \sin \theta). \end{aligned} \quad (2.12)$$

Taking the derivative of (2.7) with  $t$ , we have

$$\begin{aligned} \mathcal{X} &= p r^{p-1} \cos \theta \frac{dr}{dt} - r^p \sin \theta \frac{d\theta}{dt} \\ \mathcal{Y} &= q r^{q-1} \sin \theta \frac{dr}{dt} + r^q \cos \theta \frac{d\theta}{dt}. \end{aligned} \quad (2.13)$$

(2.13) yields that

$$\begin{aligned} \frac{dr}{dt} &= r \frac{r^q \cos \theta \mathcal{X} + r^p \sin \theta \mathcal{Y}}{r^{p+q}(p \cos^2 \theta + q \sin^2 \theta)} = \frac{r^{2pq+1}}{r^{p+q}(p \cos^2 \theta + q \sin^2 \theta)} \sum_{k=0}^{\infty} R_k(\theta) r^k, \\ \frac{d\theta}{dt} &= \frac{-qr^q \sin \theta \mathcal{X} + pr^p \cos \theta \mathcal{Y}}{r^{p+q}(p \cos^2 \theta + q \sin^2 \theta)} = \frac{r^{2pq}}{r^{p+q}(p \cos^2 \theta + q \sin^2 \theta)} \sum_{k=0}^{\infty} Q_k(\theta) r^k. \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} R_k(\theta) &= \cos \theta \mathcal{X}_{2pq-q+k}(\cos \theta, \sin \theta) + \sin \theta \mathcal{Y}_{2pq-p+k}(\cos \theta, \sin \theta), \\ Q_k(\theta) &= -q \sin \theta \mathcal{X}_{2pq-q+k}(\cos \theta, \sin \theta) + p \cos \theta \mathcal{Y}_{2pq-p+k}(\cos \theta, \sin \theta), \end{aligned} \quad (2.15)$$

Especially,

$$\begin{aligned} R_0(\theta) &= \cos \theta \sin \theta (q \cos^{2q-2} \theta - p \sin^{2p-2} \theta), \\ Q_0(\theta) &= pq (\cos^{2q} \theta + \sin^{2p} \theta) > 0. \end{aligned} \quad (2.16)$$

(2.14) yields that by transformation (2.7) system (2.10) could be rewritten as the following equation:

$$\frac{dr}{d\theta} = r \frac{\sum_{k=0}^{\infty} R_k(\theta) r^k}{\sum_{k=0}^{\infty} Q_k(\theta) r^k} = \frac{R_0(\theta)}{Q_0(\theta)} r + o(r). \quad (2.17)$$

because  $Q_0(\theta) \neq 0$ , for sufficiently small  $h$ , the solution of system (2.17) which satisfy initial condition

$$r|_{\theta=0} = h \quad (2.18)$$

is a power series of  $h$  with non-zero radius convergence when  $|\theta| < 4\pi$ . Let

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} \nu_k(\theta) h^k. \quad (2.19)$$

where

$$\nu_1(0) = 1, \quad \nu_k(0) = 0, \quad k = 2, 3, \dots. \quad (2.20)$$

Furthermore

$$\nu_1(\theta) = \exp \int_0^\theta \frac{R_0(\theta)}{Q_0(\theta)} d\theta = (\cos^{2q} \theta + \sin^{2p} \theta)^{\frac{-1}{2pq}}. \quad (2.21)$$

So the origin of system (2.10) is a focus or center, the Poincaré successive function in the neighborhood of the origin can be written as

$$\Delta(h) = \tilde{r}(2\pi, h) - h = \sum_{k=2}^{\infty} \nu_k(2\pi) h^k. \quad (2.22)$$

**Remark 2.1.** If  $p, q$  are not relatively prime, namely, there exists positive constants  $d, p^*, q^*$  which satisfy

$$p = dp^*, \quad q = dq^*, \quad (2.23)$$

where  $d > 1$ . Then the transformation (2.7) is equivalent to

$$x = (r^*)^{p^*} \cos \theta, \quad y = (r^*)^{q^*} \sin \theta \quad (2.24)$$

where

$$r^* = r^d. \quad (2.25)$$

and the system (2.17) could be changed into equation of polar coordinates by transformation (2.25).

### 3 Some prosperities of Successive function and focus values

Because the functions of right hand of system (2.17) is periodic function of  $\theta$  with period  $2\pi$ ,

**Proposition 3.1.** For sufficiently small constant  $h$ , when  $|\theta| < 4\pi$ , we have

$$\tilde{r}(\theta + 2\pi, h) = \tilde{r}(\theta, \tilde{r}(2\pi, h)). \quad (3.1)$$

**Proposition 3.2.** If  $p, q$  are prime numbers, for sufficiently small constant  $h$ , when  $|\theta| < 4\pi$ , we have

$$-\tilde{r}(\theta + \pi, h) = \tilde{r}(\theta, -\tilde{r}(\pi, h)). \quad (3.2)$$

*Proof.* Because  $p, q$  are prime numbers, the transformation (2.7) is equivalent to

$$x = (-r)^p \cos(\theta + \pi), \quad y = (-r)^q \sin(\theta + \pi). \quad (3.3)$$

It is easy to testify that system (2.17) keep formally unchanged by transformation  $r \rightarrow -r, \theta \rightarrow \theta + \pi$ , so  $r = -\tilde{r}(\theta + \pi, h)$  is a solution of (2.17) which satisfy initial condition  $r|_{\theta=0} = -\tilde{r}(\pi, h)$ .

On the other hand,  $r = \tilde{r}(\theta, -\tilde{r}(\pi, h))$  is another solution of (2.17) which satisfy the same initial condition. So we can get (3.2) easily by uniqueness theorem for the solution.  $\square$

**Proposition 3.3.** *If  $p, q$  are even number, then  $\tilde{r}(\theta, h)$  is an odd function of  $h$ .*

*Proof.* If  $p, q$  are even number, then  $X(r^p \cos \theta, r^q \sin \theta)$  and  $Y(r^p \cos \theta, r^q \sin \theta)$  are even functions of  $r$ , so the functions of right han of system (2.17) is an odd function of  $r$ , which shows that Proposition 3.2 holds.  $\square$

**Proposition 3.4.** *If  $p$  is an odd number,  $q$  is an even number, for sufficiently small constant  $h$ , when  $|\theta| < 4\pi$ , we have*

$$-\tilde{r}(\pi - \theta, h) = \tilde{r}(\theta, -\tilde{r}(\pi, h)). \quad (3.4)$$

*Proof.* Because  $p$  is an odd number,  $q$  is an even number, the transformation (2.7) is equivalent to

$$x = (-r)^p \cos(\pi - \theta), \quad y = (-r)^q \sin(\pi - \theta). \quad (3.5)$$

It is easy to testify that system (2.17) keep formally unchanged by transformation  $r \rightarrow -r$ ,  $\theta \rightarrow \pi - \theta$ , so  $r = -\tilde{r}(\pi - \theta, h)$  is a solution of (2.17) which satisfy initial condition  $r|_{\theta=0} = -\tilde{r}(\pi, h)$ . on the other hand,  $r = \tilde{r}(\theta, -\tilde{r}(\pi, h))$  is another solution of (2.17) which satisfy the same initial condition. So we can get (3.5) easily by uniqueness theorem for the solution.  $\square$

**Proposition 3.5.** *If  $p$  is a even number,  $q$  is an odd number, for sufficiently small constant  $h$ , when  $|\theta| < 4\pi$ , we have*

$$-\tilde{r}(\pi - \theta, h) = \tilde{r}(\theta, -\tilde{r}(\pi, h)). \quad (3.6)$$

*Proof.* Because  $p$  is an even number,  $q$  is an odd number, the transformation (2.7) is equivalent to

$$x = (-r)^p \cos(2\pi - \theta), \quad y = (-r)^q \sin(2\pi - \theta). \quad (3.7)$$

It is easy to testify that system (2.17) keep formally unchanged by transformation  $r \rightarrow -r$ ,  $\theta \rightarrow 2\pi - \theta$ , so  $r = -\tilde{r}(2\pi - \theta, h)$  is a solution of (2.17) which satisfy initial condition  $r|_{\theta=0} = -\tilde{r}(2\pi, h)$ . on the other hand,  $r = \tilde{r}(\theta, -\tilde{r}(2\pi, h))$  is another solution of (2.17) which satisfy the same initial condition. So we can get (3.6) easily by uniqueness theorem for the solution.  $\square$

Suppose  $f(h) = \sum_{k=0}^{\infty} c_k h^k$  is a form series of  $h$ , we denote the coefficient of  $h^m$  of  $f(h)$  by  $[f(h)]_m$ , namely,  $[f(h)]_m = c_m$ . Then

**Lemma 3.1.** Suppose  $m$  is a positive integer and greater than 1, if  $\nu_k(2\pi) = 0$  when  $1 < k < m$ , namely

$$\tilde{r}(2\pi, h) = h + \nu_m(2\pi)h^m + o(h^m), \quad (3.8)$$

then

$$[\tilde{r}^k(2\pi, h)]_m = \begin{cases} \nu_m(2\pi), & \text{if } k = 1, \\ 0, & \text{if } 1 < k < m, \\ 1, & \text{if } k = m, \\ 0, & \text{if } k > m. \end{cases} \quad (3.9)$$

**Theorem 3.1.** If  $p+q$  is an even number, when  $k > 1$ , the first subscript satisfying  $\nu_k(2\pi) \neq 0$  in  $\{\nu_k(2\pi)\}$  is an odd number.

*Proof.* If  $p$  and  $q$  are even numbers,  $\tilde{r}(\theta, h)$  is an odd function of  $h$  by Proposition 3.3, obviously, Theorem 3.1 holds

Next we suppose  $p$  and  $q$  are odd numbers. Let  $\theta = \pi$  in (3.1) and  $\theta = 2\pi$  in (3.2), then

$$\begin{aligned} \tilde{r}(3\pi, h) &= \tilde{r}(\pi, \tilde{r}(2\pi, h)), \\ -\tilde{r}(3\pi, h) &= \tilde{r}(2\pi, -\tilde{r}(\pi, h)). \end{aligned} \quad (3.10)$$

From (3.10), we have

$$\tilde{r}(\pi, \tilde{r}(2\pi, h)) + \tilde{r}(2\pi, -\tilde{r}(\pi, h)) = 0. \quad (3.11)$$

Namely

$$\sum_{k=1}^{\infty} \nu_k(\pi) \tilde{r}^k(2\pi, h) + \sum_{k=1}^{\infty} (-1)^k \nu_k(2\pi) \tilde{r}^k(\pi, h) = 0. \quad (3.12)$$

For any positive integer  $m$ , (3.12) yields that

$$\left[ \sum_{k=1}^{\infty} \nu_k(\pi) \tilde{r}^k(2\pi, h) + \sum_{k=1}^{\infty} (-1)^k \nu_k(2\pi) \tilde{r}^k(\pi, h) \right]_m = 0. \quad (3.13)$$

If Lemma 3.1 holds and  $\nu_1(\pi) = \nu_1(2\pi) = 1$ , so (3.9) shows that

$$\begin{aligned} \left[ \sum_{k=1}^{\infty} \nu_k(\pi) \tilde{r}^k(2\pi, h) \right]_m &= \nu_m(\pi) + \nu_m(2\pi), \\ \left[ \sum_{k=1}^{\infty} (-1)^k \nu_k(2\pi) \tilde{r}^k(\pi, h) \right]_m &= \left[ -\tilde{r}(\pi, h) + (-1)^m \nu_m(2\pi) h^m \right]_m \\ &= -\nu_m(\pi) + (-1)^m \nu_m(2\pi). \end{aligned} \quad (3.14)$$

Furthermore, (3.13) and (3.14) yields that

$$\left[ (-1)^m + 1 \right] \nu_m(2\pi) = 0. \quad (3.15)$$

So if  $m$  is an even number, (3.15) yields that  $\nu_m(2\pi) = 0$ , so Theorem 3.1 holds.  $\square$



**Theorem 3.2.** *If  $p$  is an odd number and  $q$  is an even number, when  $k > 1$ , the first subscript satisfying  $\nu_k(2\pi) \neq 0$  in  $\{\nu_k(2\pi)\}$  is even number.*

*Proof.* Let  $\theta = -\pi$  in (3.1), we have

$$\tilde{r}(\pi, h) = \tilde{r}(-\pi, \tilde{r}(2\pi, h)). \quad (3.16)$$

For any positive constant  $m$ ,

$$\left[ \tilde{r}(\pi, h) - \sum_{k=1}^{\infty} \nu_k(-\pi) \tilde{r}^k(2\pi, h) \right]_m = 0. \quad (3.17)$$

If Lemma 3.1 holds, (3.9) and (3.17) yield that

$$\nu_m(\pi) - \nu_m(-\pi) - \nu_m(2\pi) = 0. \quad (3.18)$$

Furthermore, let  $\theta = 2\pi$  in (3.4), we have

$$\tilde{r}(-\pi, h) + \tilde{r}(2\pi, -\tilde{r}(\pi, h)) = 0. \quad (3.19)$$

For any positive constant  $m$ ,

$$\left[ \tilde{r}(-\pi, h) + \sum_{k=1}^{\infty} (-1)^k \nu_k(2\pi) \tilde{r}^k(\pi, h) \right]_m = 0. \quad (3.20)$$

When Lemma 3.1 holds, from (3.20), we get

$$\left[ \tilde{r}(-\pi, h) - \tilde{r}(\pi, h) + (-1)^m \nu_m(2\pi) \tilde{r}^m(\pi, h) \right]_m = 0. \quad (3.21)$$

Namely

$$\nu_m(-\pi) - \nu_m(\pi) + (-1)^m \nu_m(2\pi) = 0. \quad (3.22)$$

(3.18) and (3.22) show that

$$\left[ (-1)^m - 1 \right] \nu_m(2\pi) = 0. \quad (3.23)$$

If  $m$  is an odd number, (3.23) yields that  $\nu_m(2\pi) = 0$ , so Theorem 3.2 holds.  $\square$

**Theorem 3.3.** *If  $p$  is an even number and  $q$  is an odd number, when  $k > 1$ , the first subscript satisfying  $\nu_k(2\pi) \neq 0$  in  $\{\nu_k(2\pi)\}$  is even number.*

*Proof.* If  $p$  is an even number and  $q$  is an odd number, let  $\theta = 2\pi$  in (3.6), we have

$$h + \tilde{r}(2\pi, -\tilde{r}(2\pi, h)) = 0. \quad (3.24)$$

For any positive integer  $m$ ,

$$\left[ h + \sum_{k=1}^{\infty} (-1)^k \nu_k(2\pi) \tilde{r}^k(2\pi, h) \right]_m = 0. \quad (3.25)$$

Suppose  $\nu_k(2\pi) = 0$  when  $1 < k < m$ , then (3.25) yields that

$$\left[ -\tilde{r}(2\pi, h) + (-1)^m \nu_m(2\pi) \tilde{r}^m(2\pi, h) \right]_m = 0. \quad (3.26)$$

We can get (3.23) easily by (3.26), so Theorem 3.3 holds.  $\square$

Based on Theorem 3.1, Theorem 3.2 and Theorem 3.3, we have

**Definition 3.1.** If  $p+q$  is even number,  $\nu_{2m+1}(2\pi)$  is called to be the  $m$ -th focal values of the origin of system (2.10); If  $p+q$  is odd number,  $\nu_{2m}(2\pi)$  is called to be the  $m$ -th focal values of the origin of system (2.10)  $m = 1, 2, \dots$ .

**Definition 3.2.** If  $p+q$  is even number, the first nonzero in  $\{\nu_k(2\pi)\}$  is  $\nu_{2m+1}(2\pi)$ , then the origin is called the  $m$ -order weak (fine) focus of (2.10); If  $p+q$  is odd number, the first nonzero in  $\{\nu_k(2\pi)\}$  is  $\nu_{2m}(2\pi)$ , then the origin is called the  $m$ -order weak (fine) focus of (2.10).

The definition of algebraic equivalence was given in [55] (see also [1]).

**Definition 3.3.** Suppose that  $\{\lambda_m\}$  and  $\{\tilde{\lambda}_m\}$  are polynomials of  $(a_{kj})'$ 's,  $(b_{kj})'$ 's which are coefficient of functions of right hand of system (2.8),  $\zeta_1^{(m)}, \zeta_2^{(m)}, \dots, \zeta_{m-1}^{(m)}$  are polynomials of  $(a_{kj})'$ 's,  $(b_{kj})'$ 's. If there exists a positive integer  $m$  satisfy that

$$\lambda_m = \tilde{\lambda}_m + \left( \zeta_1^{(m)} \lambda_1 + \zeta_2^{(m)} \lambda_2 + \dots + \zeta_{m-1}^{(m)} \lambda_{m-1} \right), \quad (3.27)$$

then  $\lambda_m$  and  $\tilde{\lambda}_m$  are algebraic equivalence, denote by  $\lambda_m \sim \tilde{\lambda}_m$ .

If there exists a positive integer  $m$  satisfy that  $\lambda_m \sim \tilde{\lambda}_m$ , then sequence of functions  $\{\lambda_m\}$  and  $\{\tilde{\lambda}_m\}$  is called to be algebraic equivalence, denote by  $\{\lambda_m\} \sim \{\tilde{\lambda}_m\}$ .

Definition 3.3 yields that

1) Algebraic equivalence relation of sequence of functions is reflexive, symmetric, and transitive

2) If there exists a positive integer  $m$  satisfy that  $\lambda_m \sim \tilde{\lambda}_m$ , when  $\lambda_1 = \lambda_2 = \dots = \lambda_{m-1} = 0$ ,  $\lambda_m = \tilde{\lambda}_m$ ;

3)  $\lambda_1 \sim \tilde{\lambda}_1$  means that  $\lambda_1 = \tilde{\lambda}_1$ .

From Theorem 3.1, Theorem 3.2, Theorem 3.3, we can conclude that

**Theorem 3.4.** *For system (2.8), if  $p + q$  is an even number, then  $\nu_{2m}(2\pi) \sim 0$ ; if  $p + q$  is an odd number, then  $\nu_{2m+1}(2\pi) \sim 0$ .  $m = 1, 2, \dots$ .*

Suppose that

$$g(h) = h + \sum_{k=2}^{\infty} c_k h^k \quad (3.28)$$

is a power series of  $h$  with nonzero convergence radius. For sufficiently small  $h$ , it is more convenient to solve equation (2.17) with initial conditions

$$r|_{\theta=0} = g(h) \quad (3.29)$$

than to solve equation  $g(h) = h$  sometimes, (see the progress in §3 for computing focal values of the origin of system (4.1)). The solution of (2.17) can be written as a power series of  $h$  with nonzero radius convergence when  $|\theta| < 4\pi$

$$r = r^*(\theta, h) = \sum_{k=1}^{\infty} \nu_k^*(\theta) h^k, \quad (3.30)$$

where

$$\nu_1^*(0) = 1, \quad \nu_k^*(0) = c_k, \quad k = 2, 3, \dots \quad (3.31)$$

Furthermore,

$$r = \tilde{r}(\theta, g(h)) = \sum_{k=1}^{\infty} \nu_k(\theta) g(h)^k \quad (3.32)$$

is another solution of (2.17) with initial condition (3.29), so  $r^*(\theta, h) = \tilde{r}(\theta, g(h))$  by uniqueness theory of solution. Namely

$$\sum_{k=1}^{\infty} \nu_k^*(\theta) h^k = \sum_{k=1}^{\infty} \nu_k(\theta) g(h)^k \quad (3.33)$$

(3.33) shows that

**Theorem 3.5.**

$$\nu_k^*(2\pi) - \nu_k^*(0) \sim \nu_k(2\pi), \quad k = 2, 3, \dots \quad (3.34)$$

Next, we will consider the perturbed system of system (2.10)

$$\begin{aligned} \frac{dx}{dt} &= -py^{2p-1} + \sum_{m=2pq-q+1}^{\infty} \mathcal{X}_m(x, y, \varepsilon) = \mathcal{X}(x, y, \varepsilon), \\ \frac{dy}{dt} &= qx^{2q-1} + \sum_{m=2pq-p+1}^{\infty} \mathcal{Y}_m(x, y, \varepsilon) = \mathcal{Y}(x, y, \varepsilon), \end{aligned} \quad (3.35)$$

where

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \quad (3.36)$$

is a small parameter and

$$\mathcal{X}_m(x, y, \varepsilon) = \sum_{kp+jq=m} a_{kj}(\varepsilon) x^k y^j, \quad \mathcal{Y}_m(x, y, \varepsilon) = \sum_{kp+jq=m} b_{kj}(\varepsilon) x^k y^j \quad (3.37)$$

are  $m$ -order homogeneous weight polynomial of  $x, y$  with weight  $p, q$ ,  $\mathcal{X}(x, y, \varepsilon)$ ,  $\mathcal{Y}(x, y, \varepsilon)$  are power series of  $x, y, \varepsilon$  with nonzero convergence radius. Denote the solution of system (3.35) with initial condition  $r|_{\theta=0} = h$  in generalized polar coordinate (2.7) by

$$r = \tilde{r}(\theta, h, \varepsilon) = \sum_{k=1}^{\infty} \nu_k(\theta, \varepsilon) h^k. \quad (3.38)$$

We have

**Theorem 3.6.** *Suppose  $p + q$  is an even number, if the origin of system (3.35) is a  $m$ -th weak focus when  $\varepsilon = \mathbf{0}$ , and choosing a proper parameter  $\varepsilon$  in  $|\varepsilon| \ll 1$ , we have*

$$\begin{aligned} 0 < |\nu_3(2\pi, \varepsilon)| &\ll |\nu_5(2\pi, \varepsilon)| \ll \cdots |\nu_{2m+1}(2\pi, \varepsilon)|, \\ \nu_{2k-1}(2\pi, \varepsilon) \nu_{2k+1}(2\pi, \varepsilon) &< 0, \quad k = 2, 3, \dots, m, \end{aligned} \quad (3.39)$$

then there exist  $m - 1$  limit cycles in the neighborhood of the origin of system (3.35).

**Theorem 3.7.** *Suppose  $p + q$  is an odd number, if the origin of system (3.35) is a  $m$ -th weak focus when  $\varepsilon = \mathbf{0}$ , and choosing a proper parameter  $\varepsilon$  in  $|\varepsilon| \ll 1$ , we have*

$$\begin{aligned} 0 < |\nu_2(2\pi, \varepsilon)| &\ll |\nu_4(2\pi, \varepsilon)| \ll \cdots |\nu_{2m}(2\pi, \varepsilon)|, \\ \nu_{2k}(2\pi, \varepsilon) \nu_{2k+2}(2\pi, \varepsilon) &< 0, \quad k = 1, 2, \dots, m, \end{aligned} \quad (3.40)$$

then there exist  $m - 1$  limit cycles in the neighborhood of the origin of system (3.35).

**Theorem 3.8.** *Suppose  $p + q$  is an odd number, if the origin of system (3.35) is a  $m$ -th weak focus when  $\varepsilon = \mathbf{0}$ , and the Jacobi matrix of  $\nu_3(2\pi, \varepsilon), \nu_5(2\pi, \varepsilon), \dots, \nu_{2m-1}(2\pi, \varepsilon)$  with respect to  $\varepsilon$  is*

$$J = \frac{\partial(\nu_3, \nu_5, \dots, \nu_{2m-1})}{\partial(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)} \quad (3.41)$$

There is a  $m - 1$  order determinant which do not equal to zero when  $\varepsilon = \mathbf{0}$ , then choosing a proper parameter  $\varepsilon$  in  $|\varepsilon| \ll 1$ , there exist  $m - 1$  limit cycles in the neighborhood of the origin of system (3.35).

**Theorem 3.9.** *Suppose  $p + q$  is an odd number, if the origin of system (3.27) is a  $m$ -th weak focus when  $\varepsilon = \mathbf{0}$ , and the Jacobi matrix of  $\nu_2(2\pi, \varepsilon), \nu_4(2\pi, \varepsilon), \dots, \nu_{2m-2}(2\pi, \varepsilon)$  with respect to  $\varepsilon$  is written as*

$$J = \frac{\partial(\nu_2, \nu_4, \dots, \nu_{2m-2})}{\partial(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)} \quad (3.42)$$

There is a  $m - 1$  order determinant which do not equal to zero when  $\varepsilon = \mathbf{0}$ , then choosing a proper parameter  $\varepsilon$  in  $|\varepsilon| \ll 1$ , there exist  $m - 1$  limit cycles in the neighborhood of the origin of system (3.35).

## 4 Center-focus determination and limit cycle bifurcation for $2 : 3$ homogeneous weight singular point

Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= -2y^3 + (a_{22}x^2y^2 + a_{50}x^5) = -2y^3 + \mathcal{X}_{10} = \mathcal{X}(x, y), \\ \frac{dy}{dt} &= 3x^5 + (b_{13}xy^3 + b_{41}x^4y) = 3x^5 + \mathcal{Y}_{11} = \mathcal{Y}(x, y). \end{aligned} \quad (4.1)$$

where

$$\mathcal{X}_{10} = a_{22}x^2y^2 + a_{50}x^5, \quad \mathcal{Y}_{11} = b_{13}xy^3 + b_{41}x^4y. \quad (4.2)$$

Let

$$x = r^2 \cos \theta, \quad y = r^3 \sin \theta, \quad (4.3)$$

we have

$$\frac{dr}{d\theta} = r \frac{R_0(\theta) + R_1(\theta)r}{Q_0(\theta) + Q_1(\theta)r}. \quad (4.4)$$

where

$$\begin{aligned} R_0(\theta) &= \cos \theta \sin \theta (3 \cos^4 \theta - 2 \sin^2 \theta), \quad Q_0(\theta) = 6(\cos^6 \theta + \sin^4 \theta), \\ R_1(\theta) &= \cos \theta [\cos^3 \theta (a_{50} \cos^2 \theta + b_{41} \sin^2 \theta) + \sin^2 \theta (a_{22} \cos^2 \theta + b_{13} \sin^2 \theta)], \\ Q_1(\theta) &= -\cos^2 \theta \sin \theta [(3a_{50} - 2b_{41}) \cos^3 \theta + (3a_{22} - 2b_{13}) \sin^2 \theta]. \end{aligned} \quad (4.5)$$

Denote the solution of equation (4.4) with initial condition  $r|_{\theta=0} = h$  by  $r = \sum_{k=1}^{\infty} \nu_k(\theta) h^k$ , and

$$\nu_k(\theta) = \nu_1(\theta) u_k(\theta), \quad k = 2, 3, \dots, \quad (4.6)$$

it is easy to get

$$\nu_1(\theta) = \frac{1}{(\cos^6 \theta + \sin^4 \theta)^{\frac{1}{12}}}, \quad (4.7)$$

and

$$\begin{aligned} u_2'(\theta) &= \frac{\cos \theta (2 \cos^2 \theta + 3 \sin^2 \theta)}{36(\cos^6 \theta + \sin^4 \theta)^{\frac{25}{12}}} \\ &\times (3a_{50} \cos^9 \theta + 3a_{22} \cos^6 \theta \sin^2 \theta + 2b_{41} \cos^3 \theta \sin^4 \theta + 2b_{13} \sin^6 \theta). \end{aligned} \quad (4.8)$$

(4.8) yields that

$$u_2(\theta) = \frac{-\cos^2 \theta \sin \theta (2b_{41} \cos^3 \theta + 5b_{13} \sin^2 \theta)}{60 (\cos^6 \theta + \sin^4 \theta)^{\frac{13}{12}}} + \frac{1}{24}(2a_{22} + 3b_{13})f_2(\theta) + \frac{1}{60}(5a_{50} + b_{41})g_2(\theta), \quad (4.9)$$

where

$$f_2(\theta) = \int_0^\theta \frac{\cos^7 \varphi \sin^2 \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{25}{12}}} d\varphi, \quad (4.10)$$

$$g_2(\theta) = \int_0^\theta \frac{\cos^{10} \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{25}{12}}} d\varphi.$$

(4.6) and (4.10) show the following proposition.

**Proposition 4.1.** *The first focal value of system (4.1) is*

$$\nu_2(2\pi) = \frac{1}{60}(5a_{50} + b_{41}) \int_0^{2\pi} \frac{\cos^{10} \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{25}{12}}} d\varphi \quad (4.11)$$

Suppose the first focal value of origin of system (4.1) is zero, namely,

$$5a_{50} + b_{41} = 0. \quad (4.12)$$

furthermore, we have

$$u_3(\theta) = u_2^2(\theta) + \frac{1}{480}(3a_{22} - 2b_{13})(2a_{22} + 3b_{13})f_3(\theta) + \frac{\cos^4 \theta}{50400 (\cos^6 \theta + \sin^4 \theta)^{\frac{13}{6}}} \left[ 120(2a_{22} + 3b_{13})b_{41} \cos^9 \theta - 7(90a_{22}^2 + 75a_{22}b_{13} - 90b_{13}^2 - 52b_{41}^2) \cos^6 \theta \sin^2 \theta - 40(15a_{22} - 23b_{13})b_{41} \cos^3 \theta \sin^4 \theta - 350(3a_{22} - 2b_{13})b_{13} \sin^6 \theta \right] \quad (4.13)$$

where

$$f_3(\theta) = \int_0^\theta \frac{\cos^{15} \varphi \sin \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{19}{6}}} d\varphi \quad (4.14)$$

after more computation, we get

$$u_4(\theta) = -u_2^3(\theta) + 2u_2(\theta)u_3(\theta) + G_1w_2(\theta) + G_2 + s_4f_4(\theta) + r_4g_4(\theta) \quad (4.15)$$

where

$$G_1 = \frac{(2a_{22} + 3b_{13}) \cos^4 \theta}{1209600(\cos^6 \theta + \sin^4 \theta)^{\frac{13}{6}}} \left[ (120(2a_{22} + 3b_{13})b_{41} \cos^9 \theta \right. \\ \left. - 7(90a_{22}^2 + 75a_{22}b_{13} - 90b_{13}^2 - 52b_{41}^2) \cos^6 \theta \sin^2 \theta \right. \\ \left. - 40(15a_{22} - 23b_{13})b_{41} \cos^3 \theta \sin^4 \theta - 350(3a_{22} - 2b_{13})b_{13} \sin^6 \theta \right], \quad (4.16)$$

$$G_2 = \frac{\sin^3 \theta}{117936000(\cos^6 \theta + \sin^4 \theta)^{\frac{13}{4}}} \sum_{k=0}^5 c_k \cos^{3(5-k)} \theta \sin^{2k} \theta. \\ f_4(\theta) = \int_0^\theta \frac{\cos^{15} \varphi \sin \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{19}{6}}} f_2(\theta) d\varphi \\ g_4(\theta) = \int_0^\theta \frac{\cos^{20} \varphi \sin^2 \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{17}{4}}} d\varphi, \quad (4.17) \\ s_4 = \frac{1}{11520} (3a_{22} - 2b_{13})(2a_{22} + 3b_{13})^2, \\ r_4 = \frac{-13}{16800} (5a_{22} - 3b_{13})(2a_{22} + 3b_{13})b_{41}.$$

In (4.16)

$$c_0 = 104b_{41}(5550a_{22}^2 + 4365a_{22}b_{13} - 5940b_{13}^2 - 2548b_{41}^2), \\ c_1 = 273(180a_{22}^3 + 420a_{22}^2b_{13} + 45a_{22}b_{13}^2 - 270b_{13}^3 + 4104a_{22}b_{41}^2 - 1124b_{13}b_{41}^2), \\ c_2 = -54600(10a_{22}^2 - 33a_{22}b_{13} + 19b_{13}^2)b_{41}, \\ c_3 = 182(3240a_{22}^3 - 3690a_{22}^2b_{13} + 1935a_{22}b_{13}^2 - 610b_{13}^3 + 1872a_{22}b_{41}^2 + 2808b_{13}b_{41}^2), \\ c_4 = 0, \\ c_5 = 504(2a_{22} + 3b_{13})(180a_{22}^2 - 225a_{22}b_{13} + 70b_{13}^2 + 104b_{41}^2). \quad (4.18)$$

Above computation show that

**Proposition 4.2.** *when  $\nu_2(2\pi) = 0$ , the second focal value at origin of system (4.1) is*

$$\nu_4(2\pi) = \frac{-13}{16800} (5a_{22} - 3b_{13})(2a_{22} + 3b_{13})b_{41} \\ \times \int_0^{2\pi} \frac{\cos^{20} \varphi \sin^2 \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{17}{4}}} d\varphi. \quad (4.19)$$

The same method could be used to get  $u_5(\theta), u_6(\theta)$ , then

**Proposition 4.3.** *When  $\nu_2(2\pi) = \nu_4(2\pi) = 0$ , the third focal value at origin of system (4.1)*

is

$$\nu_6(2\pi) = \frac{(2a_{22} + 3b_{13})^2 b_{41}^3}{412356420000} (575803A - 11848200B), \quad (4.20)$$

where

$$\begin{aligned} A &= 575803 \int_0^{2\pi} \frac{\cos^{36} \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{77}{12}}} d\varphi \\ B &= 11848200 \int_0^{2\pi} \frac{\cos^{28} \varphi \sin \varphi (2 \cos^2 \varphi + 3 \sin^2 \varphi)}{(\cos^6 \varphi + \sin^4 \varphi)^{\frac{16}{3}}} f_2(\varphi) d\varphi \end{aligned} \quad (4.21)$$

With the aid of computer, we have

$$575803A - 11848200B = 814653.251446 \dots > 0, \quad (4.22)$$

From Proposition 4.1~Proposition 4.3, we can conclude that

**Theorem 4.1.** *The first three (4.1) focal values at origin of system (4.1) are*

$$\begin{aligned} V_2 &= 5a_{50} + b_{41}, \\ V_4 &= -(5a_{22} - 3b_{13})(2a_{22} + 3b_{13})b_{41}, \\ V_6 &= (2a_{22} + 3b_{13})^2 b_{41}^3. \end{aligned} \quad (4.23)$$

**Theorem 4.2.** *The origin of system (4.1) is a center if and only if one of the following conditions holds:*

$$\begin{aligned} C_1 : \quad & a_{50} = -\frac{1}{5}b_{41}, \quad a_{22} = -\frac{3}{2}b_{13}; \\ C_2 : \quad & a_{50} = -\frac{1}{5}b_{41}, \quad b_{41} = 0. \end{aligned} \quad (4.24)$$

*Proof.* Theorem 4.1 yields that necessary condition holds. On the other hand, if condition  $C_1$  in Theorem 4.2 holds, system (4.1) is Hamilton; if condition  $C_2$  in Theorem 4.2 holds, system (4.1) is symmetric with  $x$  axis, so the sufficient condition hold.  $\square$

Suppose the coefficients of functions at right hand of system (4.1) are

$$b_{41} = 1, \quad a_{50} = -\frac{1}{5}(1 - \varepsilon_1), \quad a_{22} = \frac{1}{35}(5 + 7\varepsilon_2), \quad b_{13} = \frac{5}{21}, \quad (4.25)$$

Theorem 4.1 shows that the first three (4.1) focal values at origin of system (4.1) are

$$V_2 = \varepsilon_1, \quad V_4|_{\varepsilon_1=0} = -\varepsilon_2 + o(\varepsilon_2), \quad V_6|_{\varepsilon_1=\varepsilon_2=0} = 1, \quad (4.26)$$

Furthermore, Theorem 3.5 yields that

**Theorem 4.3.** *If (4.25) holds, the origin of system (4.1) is an unstable 3-th weak focus when  $0 < \varepsilon_1 \ll \varepsilon_2 \ll 1$ , there exist two limit cycles in the neighborhood of origin of system (4.1).*



Meanwhile, the origin of system (4.1) is a high order singular point, the multiple degree of higher order singular point was defined in [52] The origin of (4.1) could be broken into some singular point with lower multiple degree, and limit cycles could be bifurcated from the new singular point. Similar problem have been investigated in [53],[54]. We consider a perturbed system of system (4.1)

$$\begin{aligned}\frac{dx}{dt} &= -\delta_0\sigma^8x - \sigma^6y + \frac{1}{2}(5a_{50} + b_{41} + 4\delta_1 + 8\delta_2)\sigma^2xy^2 - 2y^3 + x^2(a_{50}x^3 + a_{22}y^2), \\ \frac{dy}{dt} &= \sigma^8x - \delta_0\sigma^8y - \frac{1}{2}(5a_{50} + b_{41} - 4\delta_1 + 8\delta_2)\sigma^4x^2y + 3x^5 + xy(b_{41}x^3 + b_{13}y^2),\end{aligned}\quad (4.27)$$

where  $\delta_0, \delta_1, \delta_2$  and  $\sigma$  are small parameters.

System (4.27) is system (4.1) when  $\sigma = 0$ .

If  $\sigma \neq 0$ , the origin of system (4.27) is an elementary focus. In order to study its Hopf bifurcation, system (4.27) can be transformed into system

$$\begin{aligned}\frac{d\xi}{d\tau} &= -\delta_0\sigma\xi - \eta + \frac{1}{2}(5a_{50} + b_{41} + 4\delta_1 + 8\delta_2)\sigma\xi\eta^2 - 2\eta^3 + \sigma\xi^2(a_{50}\xi^3 + a_{22}\eta^2), \\ \frac{d\eta}{d\tau} &= \xi - \delta_0\sigma\eta - \frac{1}{2}(5a_{50} + b_{41} - 4\delta_1 + 8\delta_2)\sigma\xi^2\eta + 3\xi^5 + \sigma\xi\eta(b_{41}\xi^3 + b_{13}\eta^2).\end{aligned}\quad (4.28)$$

by transformation

$$x = \sigma^2\xi, \quad y = \sigma^3\eta, \quad d\tau = \sigma^7dt. \quad (4.29)$$

Furthermore, we can give that

**Theorem 4.4.** *The divergence at origin of system (4.28) is  $\lambda_0 = -2\delta_0\sigma$ . When  $\delta_0 = 0$ , the first three (4.28) focal values at origin of system (4.28) are*

$$\lambda_1 = \delta_1\sigma, \quad \lambda_2|_{\delta_1=0} = -\delta_2\sigma, \quad \lambda_3|_{\delta_1=\delta_2=0} = \frac{47}{128}(5a_{50} + b_{41})\sigma. \quad (4.30)$$

Theorem 4.3 and Theorem 4.4 show that

**Theorem 4.5.** *If (4.25) holds, when*

$$0 < |\sigma| \ll 1, \quad 0 < \delta_0 \ll \delta_1 \ll \delta_2 \ll \varepsilon_1 \ll \varepsilon_2 \ll 1, \quad (4.31)$$

*there exist five limit cycles in the neighborhood of origin of system (4.27).*

## References

- [1] Yirong Liu, Jibin Li, Wentao Huang, Planar Dynamical System, Science Press and Walter de Gruyter GmbH, Berlin/Boston.

- [2] N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, *Math. Sbor.* 30 (1952) 181-196.
- [3] H. Zoladek, Eleven small limit cycles in a cubic vector field, *Nonlinearity*, 8 (1995) 843-860.
- [4] C. Christopher, Estimating limit cycle bifurcations from centers. *Differential equations with symbolic computation*, Trends Math., Birkh?user, Basel, (2005) 23-35.
- [5] P. Yu, Y. Tian, Twelve limit cycles around a singular point in a planar cubic-degree polynomial system, *Commun. Nonlinear Sci. Numer. Simulat.* 19 (2014) 2690-2705.
- [6] H. Giacomini, J. Giné, J. Llibre, The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems, *J. Differential Equations* 227 (2006) 406-426.
- [7] V. Mañosa, On the center problem for degenerate singular points of planar vector fields, *Int. J. Bifurcation and Chaos*, 2002, 12(04): 687-707.
- [8] A. Gasull, V. Manosa, F. Manosas, Monodromy and stability of a class of degenerate planar critical points, *J. Differential Equations* 182 (2002) 169-190.
- [9] J. Giné, Sufficient conditions for a center at a completely degenerate critical point, *Int. J. Bifurcation and Chaos* 12 (2002) 1659-1666.
- [10] J. Giné, J. Llibre, On the center conditions for analytic monodromic degenerate singularities, *Int. J. Bifur. Chaos Appl. Sci. Engrg.* 22 (2012) no. 12, 1250303.
- [11] J. Giné, On the degenerate center problem, *Int. J. Bifur. Chaos Appl. Sci. Engrg.* 21 (2011) no. 5, 1383-1392.
- [12] A. Gasull, J. Torregrosa, A new algorithm for the computation of the Lyapunov constants for some degenerated critical points, *Nonlinear Anal. TMA* 47 (2001) 4479-4490.
- [13] M.J. Álvarez, A. Gasull, Generating limit cycles from a nilpotent critical point via normal forms, *J. Math. Anal. Appl.* 318 (2006) 271-287.
- [14] H. Chen, Y. Liu, X. Zeng, Center conditions and bifurcation of limit cycles at degenerate singular points in a quintic polynomial differential system, *Bull. des sci. math.* 129 (2005) 127-138.
- [15] Q. Zhang, W. Gui, Y. Liu, The generalized center problem of degenerate resonant singular point, *Bull. des sci. math.* 133 (2009) 198-204.

- [16] Y. Liu, J. Li, Some Classical Problems about Planar Vector Fields (in chinese), Science press, Beijing, 2010:279-316.
- [17] M. Han, P. Yu, Normal Forms, Melnikov Functions and Bifurcations of Limit Cycles, Springer press, 2012.
- [18] Y. Liu, Theory of center-focus in a class of high order singular points and infinity, Sci. in China 31 (2001) 37-48.
- [19] A. Algaba, E. Freire, E. Gamero, C. Garca, Monodromy, center-focus and integrability problems for quasi-homogeneous polynomial systems, Nonlinear Anal. 72 (2010) 1726C1736.
- [20] A. Algaba, N. Fuentes, C. Garca, Centers of quasi-homogeneous polynomial planar systems, Nonlinear Anal. Real World Appl. 13 (2012) 419C431.
- [21] A. Algaba, E. Gamero, C. Garca, The integrability problem for a class of planar systems, Nonlinearity 22 (2009) 395C420.
- [22] A. Algaba, C. Garca, M. Reyes, Integrability of two dimensional quasi-homogeneous polynomial differential systems, Rocky Mountain J. Math. 41 (2011) 1C22.
- [23] A. Algaba, C. Garca, M. Reyes, Rational integrability of two-dimensional quasi-homogeneous polynomial differential systems, Nonlinear Anal. 73 (2010) 1318C1327.
- [24] A. Algaba, C. Garca, M.A. Teixeira, Reversibility and quasi-homogeneous normal forms of vector fields, Nonlinear Anal. 73 (2010) 510C525.
- [25] J. Argem, Sur les points singuliers multiples de systmes dynamiques dans  $\mathbb{R}^2$ , Ann. Mat. Pura Appl. (4) 79 (1968) 35C70.
- [26] L. Cair, J. Llibre, Polynomial first integrals for weight-homogeneous planar polynomial differential systems of weight degree 3, J. Math. Anal. Appl. 331 (2007) 1284C1298.
- [27] A. Cima, J. Llibre, Algebraic and topological classification of the homogeneous cubic systems in the plane, J. Math. Anal. Appl. 147 (1990) 420C448.
- [28] C.B. Collins, Algebraic classification of homogeneous polynomial vector fields in the plane, Japan J. Indust. Appl. Math. 13 (1996) 63C91.
- [29] T. Date, Classification and analysis of two-dimensional homogeneous quadratic differential equations systems, J. Differential Equations 32 (1979) 311C334.

- [30] I. Garca, On the integrability of quasihomogeneous and related planar vector fields, *Internat. J. Bifur. Chaos* 13 (2003) 995C1002.
- [31] A. Goriely, Integrability, partial integrability, and nonintegrability for systems of ordinary differential equations, *J. Math. Phys.* 37 (1996) 1871C1893.
- [32] A. Goriely, Integrability and Nonintegrability of Dynamical Systems, *Adv. Ser. Nonlinear Dynam.*, vol. 19, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
- [33] Y. Hu, On the integrability of quasihomogeneous systems and quasidegenerate infinity systems, *Adv. Difference Equ.* 2007 (2007), Art ID. 98427, 10 pp.
- [34] N.A. Korol, The integral curves of a certain differential equation, *Minsk. Gos. Ped. Inst.* Minsk (1973) 47C51 (in Russian).
- [35] J. Llibre, J.S. Prez del Ro, J.A. Rodriguez, Structural stability of planar homogeneous polynomial vector fields. Applications to critical points and to infinity, *J. Differential Equations* 125 (1996) 490C520.
- [36] J. Llibre, C. Pessoa, On the centers of the weight-homogeneous polynomial vector fields on the plane, *J. Math. Anal. Appl.* 359 (2009) 722C730.
- [37] W. Li, J. Llibre, J. Yang, Z. Zhang, Limit cycles bifurcating from the period annulus of quasi-homogeneous centers, *J. Dynam. Differential Equations* 21 (2009) 133C152.
- [38] M.H. Liu, K.Y. Guan, Reduction of quasi-homogeneous autonomous systems and reduced Kovalevskaya exponent, *Acta Math. Appl. Sin.* 31 (2008) 729C743 (in Chinese).
- [39] J. Llibre, X. Zhang, Polynomial first integrals for quasi-homogeneous polynomial differential systems, *Nonlinearity* 15 (2002) 1269C1280.
- [40] L.S. Lyagina, The integral curves of the equation, *Uspekhi Mat. Nauk* 6 (2(42)) (1951) 171C183 (in Russian).
- [41] L. Markus, Quadratic differential equations and non-associative algebras, in: *Ann. of Math. Stud.*, vol. 45, Princeton University Press, 1960, pp. 185C213.
- [42] T.A. Newton, Two dimensional homogeneous quadratic differential systems, *SIAM Rev.* 20 (1978) 120C138.
- [43] K.S. Sibirskii, N.I. Vulpe, Geometric classification of quadratic differential systems, *Differ. Equ.* 13 (1977) 548C556.

- [44] A. Tsygvintsev, On the existence of polynomial first integrals of quadratic homogeneous systems of ordinary differential equations, J. Phys. A 34 (2001) 2185C2193.
- [45] E.V. Vdovina, Classification of singular points of the equation by Forsters method, Differ. Uravn. 20 (1984) 1809C1813 (in Russian).
- [46] H. Yoshida, Necessary conditions for existence of algebraic first integrals I and II, Celestial Mech. 31 (1983) 363C379 and 381C399.
- [47] H. Yoshida, A note on Kowalevski exponents and the non-existence of an additional analytic integral, Celestial Mech. 44 (1988) 313C316.
- [48] H. Yoshida, A criterion for the non-existence of an additional analytic integral in Hamiltonian systems with  $n$  degrees of freedom, Phys. Lett. A 141 (1989) 108C112.
- [49] y. Xiong, M. Han, Planar quasi-homogeneous polynomial system with a given weight degree, Discrete and continuous dynamical system, 36(2016) 4015-4025.
- [50] y. Xiong, M. Han, Y.wang Center Problems and Limit Cycle Bifurcations in a Class of Quasi-Homogeneous Systems, International Journal of Bifurcation and Chaos, 25(2015).
- [51] Y.Zhao Limit cycles for planar semi-quasi-homogeneous polynomial vector fields, J. Math. Anal. Appl. 397 (2013) 276C284.
- [52] Liu Yirong, Multiplicity of higher order singular point of differential autonomous system, J. Cent. South Univ. Technol., **30**(3), 325-326.
- [53] Yirong Liu, Jibin Li, New study on the center problem and bifurcations of limit cycles for the Lyapunov system (1), International Journal of Bifurcation and Chaos, Vol. 19, No. 11 (2009) 3791-3801.
- [54] Yirong Liu, Feng Li, Double bifurcation of nilpotent focus, International Journal of Bifurcation and Chaos, Vol. 25, No. 3 (2015) 1550036 (10 pages), DOI: 10.1142/S0218127415500364.
- [55] Yirong Liu, Theory of center-focus in a class of higher-degree critical points at infinite points, Science in china, (Series A), 44(3), 2001, 365-377.